Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Linear Algebra and its Applications 404 (2005) 297–304
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

Improved bounds for the largest eigenvalue of trees ☆

Oscar Rojo [*,1]

Departamento de Matemáticas, Universidad Católica del Norte, Casilla 1280, Antofagasta, Chile

Received 21 November 2004; accepted 23 February 2005
Available online 12 April 2005
Submitted by R.A. Brualdi

Abstract

Let $\mathscr{T}$ be a tree with vertex set V. Let d_v denotes the degree of $v \in V$. Let $\Delta = \max\{d_v : v \in V\}$. Let $u \in V$ such that $d_u = \Delta$. Let $k = e_u + 1$ where e_u is the excentricity of u. For $j = 1, 2, \ldots, k-2$, let

$$\delta_j = \max\{d_v : \operatorname{dist}(v, u) = j\}.$$

We prove that

$$\mu_1(\mathscr{T}) < \max\left\{\max_{2\leqslant j\leqslant k-2}\left\{\sqrt{\delta_j - 1} + \delta_j + \sqrt{\delta_{j-1} - 1}\right\}, \sqrt{\delta_1 - 1} + \delta_1 + \sqrt{\Delta}, \Delta + \sqrt{\Delta}\right\}$$

and

$$\lambda_1(\mathscr{T}) < \max\left\{\max_{2\leqslant j\leqslant k-2}\left\{\sqrt{\delta_j - 1} + \sqrt{\delta_{j-1} - 1}\right\}, \sqrt{\delta_1 - 1} + \sqrt{\Delta}\right\},$$

where $\mu_1(\mathscr{T})$ and $\lambda_1(\mathscr{T})$ are the largest eigenvalue of the Laplacian matrix and adjacency matrix of T, respectively. These bounds give better results than those obtained in

☆ Work supported by Project Mecesup UCN0202 and Fondecyt 1040218, Chile.

* Tel.: +56 55 355593; fax: +56 55 355599.
E-mail address: orojo@socompa.ucn.cl
1 This research was conducted while the author was visitor at the Universidad Carlos III, Madrid, Spain.

doi:10.1016/j.laa.2005.02.023

[D. Stevanović, Bounding the largest eigenvalue of trees in terms of the largest vertex degree, Linear Algebra Appl. 360 (2003) 35–42] except if $\delta_1 = \Delta$.

AMS classification: 5C50; 15A48; 05C05

Keywords: Tree; Laplacian matrix; Adjacency matrix; Largest eigenvalue

1. Introduction

Let $\mathscr{G} = (V, E)$ be a simple undirected graph. Let $A(\mathscr{G})$ be the adjacency matrix of $\mathscr{G}$ and let $D(\mathscr{G})$ be the diagonal matrix of vertex degrees. The Laplacian matrix of $\mathscr{G}$ is $L(\mathscr{G}) = D(\mathscr{G}) - A(\mathscr{G})$. Both $A(\mathscr{G})$ and $L(\mathscr{G})$ are real symmetric matrices. Let $\mu_1(\mathscr{G})$ and $\lambda_1(\mathscr{G})$ be the largest eigenvalue of $L(\mathscr{G})$ and $A(\mathscr{G})$, respectively.

Let d_v denotes the degree of $v \in V$. The distance $d(v, u)$ from a vertex v to a vertex u is the length of the shortest path from v and u. We recall that the excentricity of a vertex u is the largest distance from u to any other vertex of the graph. Let

$$\Delta = \max \{d_v : v \in V\}.$$

It is well known that for any graph $\mathscr{G}$

$$\sqrt{\Delta} \leqslant \lambda_1 (\mathscr{G}) \leqslant \Delta$$

and

$$\Delta + 1 \leqslant \mu_1 (\mathscr{G}) \leqslant 2\Delta.$$

Recently, in [1, Theorem 1, p. 36], Stevanović proved that for a tree $\mathscr{T}$ with largest vertex degree Δ,

$$\mu_1 (\mathscr{T}) < \Delta + 2\sqrt{\Delta - 1} \tag{1}$$

and

$$\lambda_1 (\mathscr{T}) < 2\sqrt{\Delta - 1}. \tag{2}$$

Let $\mathscr{T}_k$ a rooted tree of k levels such that in each level the vertices have equal degree. For $j = 1, 2, 3, \ldots, k$, let us denote by d_{k-j+1} the degree of the vertices in level j. Observe that d_k is the degree of the root vertex and $d_1 = 1$ is the degree of the vertices in the level k.

For $j = 1, 2, 3, \ldots, k - 1$, let L_j be the $j \times j$ leading principal submatrix of the $k \times k$ symmetric tridiagonal matrix

$$L_k = \begin{bmatrix} 1 & \sqrt{d_2-1} & 0 & \cdots & \cdots & 0 \\ \sqrt{d_2-1} & d_2 & \sqrt{d_3-1} & \ddots & & \vdots \\ 0 & \sqrt{d_3-1} & d_3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \sqrt{d_{k-1}-1} & 0 \\ \vdots & & \ddots & \sqrt{d_{k-1}-1} & d_{k-1} & \sqrt{d_k} \\ 0 & \cdots & \cdots & 0 & \sqrt{d_k} & d_k \end{bmatrix} \tag{3}$$

and let A_j be the $j \times j$ leading principal submatrix of the $k \times k$ symmetric tridiagonal matrix

$$A_k = \begin{bmatrix} 0 & \sqrt{d_2-1} & 0 & \cdots & \cdots & 0 \\ \sqrt{d_2-1} & 0 & \sqrt{d_3-1} & \ddots & & \vdots \\ 0 & \sqrt{d_3-1} & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \sqrt{d_{k-1}-1} & 0 \\ \vdots & & \ddots & \sqrt{d_{k-1}-1} & 0 & \sqrt{d_k} \\ 0 & \cdots & \cdots & 0 & \sqrt{d_k} & 0 \end{bmatrix}. \tag{4}$$

Let

$$\Phi = \{1, 2, 3, \ldots, k-1\},$$
$$\Omega = \left\{ j \in \Phi : n_j > n_{j+1} \right\}$$

and let us denote by $\sigma(M)$ the spectrum of a matrix M. In [2], we proved that the eigenvalues of the Laplacian matrix and of the adjacency matrix of $\mathcal{T}_k$ are the eigenvalues of principal submatrices of the nonnegative symmetric tridiagonal matrices L_k and A_k defined above. More precisely

Theorem 1

$$\sigma\left(L\left(\mathcal{T}_k\right)\right) = \left(\bigcup_{j \in \Omega} \sigma\left(L_j\right) \right) \cup \sigma\left(L_k\right)$$

and

$$\sigma\left(A\left(\mathcal{T}_k\right)\right) = \left(\bigcup_{j \in \Omega} \sigma\left(A_j\right) \right) \cup \sigma\left(A_k\right).$$

Corollary 2

$$\mu_1 (\mathscr{T}_k) \in \sigma (L_k) \tag{5}$$

and

$$\lambda_1 (\mathscr{T}_k) \in \sigma (A_k) . \tag{6}$$

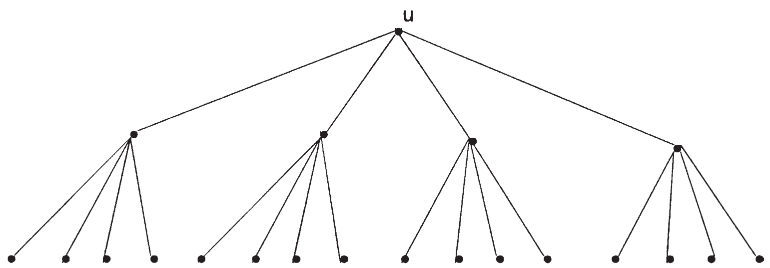
Proof. Eqs. (5) and (6) follow from Theorem 1 and from the interlacing property of the eigenvalues of Hermitian matrices. □

The use of Corollary 2 will allow us to derive new upper bounds for $\mu_1(\mathscr{T})$ and $\lambda_1(\mathscr{T})$ in terms of the vertex degrees.

2. The new upper bounds

We begin this section proving that the bounds of Stevanović given in (1) and (2) are immediate consequences of Corollary 2. We recall the notion of a *rooted Bethe tree* $B_{\Delta,k}$ [3]. The tree $B_{\Delta,1}$ is a single vertex. For $k > 1$ the tree $B_{\Delta,k}$ consists of a vertex u which is joined by edges to the roots of each of $\Delta - 1$ copies of $B_{\Delta,k-1}$. The vertex u is the root of $B_{\Delta,k}$.

Example 1. The tree $B_{5,3}$ is



We see $B_{5,3}$ is a tree of 3 levels in which the vertex root has degree equal to 4, the vertices in level 2 have degree equal to 5 and the vertices in level 3 have degree equal to 1.

In general, $B_{\Delta,k}$ is a rooted tree of k levels in which the root vertex has degree equal to $\Delta - 1$, the vertices in level j $(2 \leqslant j \leqslant k - 1)$ have degree equal to Δ and the vertices in level k have degree equal to 1. From Corollary 2, it follows that:

$$\mu_1 \left(B_{\Delta,k}\right) \in \sigma (L_k)$$

and

$$\lambda_1 \left(B_{\Delta,k}\right) \in \sigma (A_k) ,$$

where L_k and A_k are $k \times k$ matrices given by

$$L_k = \begin{bmatrix} 1 & \sqrt{\Delta-1} & 0 & \cdots & \cdots & 0 \\ \sqrt{\Delta-1} & \Delta & \sqrt{\Delta-1} & \ddots & & \vdots \\ 0 & \sqrt{\Delta-1} & \Delta & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \sqrt{\Delta-1} & 0 \\ \vdots & & \ddots & \sqrt{\Delta-1} & \Delta & \sqrt{\Delta-1} \\ 0 & \cdots & \cdots & 0 & \sqrt{\Delta-1} & \Delta-1 \end{bmatrix}$$

and

$$A_k = \begin{bmatrix} 0 & \sqrt{\Delta-1} & 0 & \cdots & \cdots & 0 \\ \sqrt{\Delta-1} & 0 & \sqrt{\Delta-1} & \ddots & & \vdots \\ 0 & \sqrt{\Delta-1} & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \sqrt{\Delta-1} & 0 \\ \vdots & & \ddots & \sqrt{\Delta-1} & 0 & \sqrt{\Delta-1} \\ 0 & \cdots & \cdots & 0 & \sqrt{\Delta-1} & 0 \end{bmatrix}.$$

Now, from the Gershgorin theorem, we obtain

$$\mu_1\left(B_{\Delta,k}\right) < \Delta + 2\sqrt{\Delta-1} \tag{7}$$

and

$$\lambda_1\left(B_{\Delta,k}\right) < 2\sqrt{\Delta-1}. \tag{8}$$

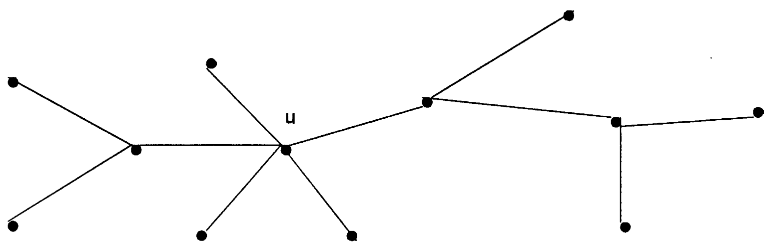
These inequalities are Lemmas 1 and 2 in [1]. Stevanović observes [1, p. 37] that for each tree $\mathscr{T}$ with largest vertex degree Δ there exists k such that $\mathscr{T}$ is an induced subgraph of $B_{\Delta,k}$. Since $\mu_1(\mathscr{T}) \leqslant \mu_1(B_{\Delta,k})$ and $\lambda_1(\mathscr{T}) \leqslant \lambda_1(B_{\Delta,k})$, the upper bounds in (1) and (2) follow from (7) and (8), respectively.

Let $\mathscr{T}$ be a tree with largest vertex degree Δ. Let u be a vertex of $\mathscr{T}$ with degree $d_u = \Delta$. Let $k = e_u + 1$ where e_u is the excentricity of u. For $j = 1, 2, \ldots, k-1$, let

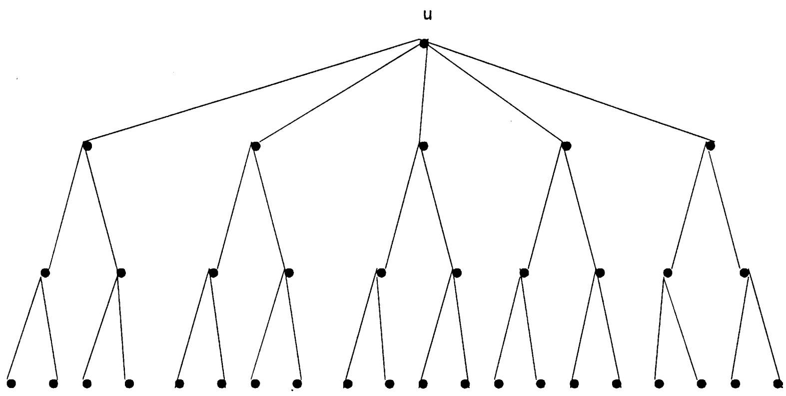
$$\delta_j = \max\left\{d_v : \operatorname{dist}(v, u) = j\right\}.$$

Observe that $\delta_{k-1} = 1$. Let $\mathscr{T}_k$ be the rooted tree of k levels in which u is the root vertex and such that, for $j = 2, 3, \ldots, k$, the vertices in level j have degree δ_{j-1}. Clearly, $\mathscr{T}$ is an induced subgraph of $\mathscr{T}_k$.

Example 2. Let $\mathscr{T}$ be the tree



Then $\Delta = 5$, $e_u = 3$, $k = 4$, $\delta_1 = 3$, $\delta_2 = 3$, $\delta_3 = 1$. The corresponding tree $\mathscr{T}_4$ is



From Corollary 2

$$\mu_1(\mathscr{T}_4) \in \sigma(L_4) \quad \text{and} \quad \lambda_1(\mathscr{T}_4) \in \sigma(A_4),$$

where

$$L_4 = \begin{bmatrix} 1 & \sqrt{2} & 0 & 0 \\ \sqrt{2} & 3 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 3 & \sqrt{5} \\ 0 & 0 & \sqrt{5} & 5 \end{bmatrix}$$

and

$$A_4 = \begin{bmatrix} 0 & \sqrt{2} & 0 & 0 \\ \sqrt{2} & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{5} \\ 0 & 0 & \sqrt{5} & 0 \end{bmatrix}.$$

Hence

$$\mu_1(\mathscr{T}) \leqslant \mu_1(\mathscr{T}_4) < 5 + \sqrt{5}$$

and

$$\lambda_1(\mathscr{T}) \leqslant \lambda_1(\mathscr{T}_4) < \sqrt{2} + \sqrt{5}.$$

For the tree $\mathscr{T}$ of this example, (1) and (2) give the bounds

$$\mu_1(\mathscr{T}) < 5 + 2\sqrt{4} = 9$$

and

$$\lambda_1(\mathscr{T}) < 2\sqrt{4} = 4.$$

Theorem 3. *Let $\mathscr{T}$ be a tree with largest vertex degree Δ. Let u be a vertex of $\mathscr{T}$ such that $d_u = \Delta$. Let $k = e_u + 1$ where e_u is the excentricity of u. For $j = 1, 2, \ldots, k-1$, let*

$$\delta_j = \max\{d_v : \operatorname{dist}(v, u) = j\}.$$

Then

$$\mu_1(\mathscr{T}) < \max\left\{\max_{2\leqslant j\leqslant k-2}\left\{\sqrt{\delta_j - 1} + \delta_j + \sqrt{\delta_{j-1} - 1}\right\},\right. \left.\sqrt{\delta_1 - 1} + \delta_1 + \sqrt{\Delta}, \Delta + \sqrt{\Delta}\right\} \tag{9}$$

and

$$\lambda_1(\mathscr{T}) < \max\left\{\max_{2\leqslant j\leqslant k-2}\left\{\sqrt{\delta_j - 1} + \sqrt{\delta_{j-1} - 1}\right\}, \sqrt{\delta_1 - 1} + \sqrt{\Delta}\right\}. \tag{10}$$

Proof. Let $\mathscr{T}_k$ be the rooted tree of k levels in which u is the root vertex and such that, for $j = 2, 3, \ldots, k$, the vertices in level j have degree δ_{j-1}. Then, $\mathscr{T}$ is an induced subgraph of $\mathscr{T}_k$. We apply Corollary 2 to obtain

$$\mu_1(L(\mathscr{T}_k)) \in L_k$$

and

$$\mu_1(A(\mathscr{T}_k)) \in A_k,$$

where L_k and A_k are the $k \times k$ matrices given by

$$L_k = \begin{bmatrix} 1 & \sqrt{\delta_{k-2} - 1} & 0 & \cdots & \cdots & 0 \\ \sqrt{\delta_{k-2} - 1} & \delta_{k-2} & \sqrt{\delta_{k-3} - 1} & \ddots & & \vdots \\ 0 & \sqrt{\delta_{k-3} - 1} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \delta_2 & \sqrt{\delta_1 - 1} & 0 \\ \vdots & & \ddots & \sqrt{\delta_1 - 1} & \delta_1 & \sqrt{\Delta} \\ 0 & \cdots & \cdots & 0 & \sqrt{\Delta} & \Delta \end{bmatrix}$$

and

$$A_k = \begin{bmatrix} 0 & \sqrt{\delta_{k-2}-1} & 0 & \cdots & \cdots & 0 \\ \sqrt{\delta_{k-2}-1} & 0 & \sqrt{\delta_{k-3}-1} & \ddots & & \vdots \\ 0 & \sqrt{\delta_{k-3}-1} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \sqrt{\delta_1-1} & 0 \\ \vdots & & \ddots & \sqrt{\delta_1-1} & 0 & \sqrt{\Delta} \\ 0 & \cdots & \cdots & 0 & \sqrt{\Delta} & 0 \end{bmatrix}.$$

We now apply Gershgorin theorem to conclude

$$\mu_1(\mathscr{T}_k) < \max\left\{ \max_{2\leqslant j\leqslant k-2}\left\{\sqrt{\delta_j-1}+\delta_j+\sqrt{\delta_{j-1}-1}\right\}, \sqrt{\delta_1-1}+\delta_1+\sqrt{\Delta}, \Delta+\sqrt{\Delta}\right\}$$

and

$$\lambda_1(\mathscr{T}_k) < \max\left\{ \max_{2\leqslant j\leqslant k-2}\left\{\sqrt{\delta_j-1}+\sqrt{\delta_{j-1}-1}\right\}, \sqrt{\delta_1-1}+\sqrt{\Delta}\right\}.$$

Since $\mathscr{T}$ is an induced subgraph of $\mathscr{T}_k$, $\mu_1(\mathscr{T}) \leqslant \mu_1(\mathscr{T}_k)$ and $\lambda_1(\mathscr{T}) \leqslant \lambda_1(\mathscr{T}_k)$. Thus the upper bounds (9) and (10) follow. □

Finally, we observe that

$$\begin{aligned} &\sqrt{\delta_j-1}+\delta_j+\sqrt{\delta_{j-1}-1} \leqslant \Delta+2\sqrt{\Delta-1} && \text{for } j=2,3,\ldots,k-2, \\ &\sqrt{\delta_1-1}+\delta_1+\sqrt{\Delta} \leqslant \Delta+2\sqrt{\Delta-1} && \text{except if } \delta_1=\Delta \end{aligned}$$

and

$$\begin{aligned} &\sqrt{\delta_j-1}+\sqrt{\delta_{j-1}-1} \leqslant 2\sqrt{\Delta-1} && \text{for } j=2,3,\ldots,k-2, \\ &\sqrt{\delta_1-1}+\sqrt{\Delta} \leqslant 2\sqrt{\Delta-1} && \text{except if } \delta_1=\Delta. \end{aligned}$$

Consequently, the new bounds (9) and (10) give better results than the Stevanović bounds (1) and (2) except if $\delta_1 = \Delta$.

References

[1] D. Stevanović, Bounding the largest eigenvalue of trees in terms of the largest vertex degree, Linear Algebra Appl. 360 (2003) 35–42.
[2] O. Rojo, R. Soto, The spectra of the adjacency matrix and Laplacian matrix for some balanced trees, Linear Algebra Appl., in press (doi:10.1016/j.laa.2005.01.011).
[3] O.J. Heilmann, E.H. Lieb, Theory of monomer–dimer systems, Comm. Math. Phys. 25 (1972) 190–232 .